Centroid

The centroid is a geometric property of a line, surface, or volume, and represents the central point of the line, area, or body.

A central point is that point about which the summation of the first moments of the elements of the body results in zero.

Center of Gravity (C.G.)

Center of gravity of a body is a point through which the resultant of the distributed gravity forces acts irrespective of the orientation of the body.

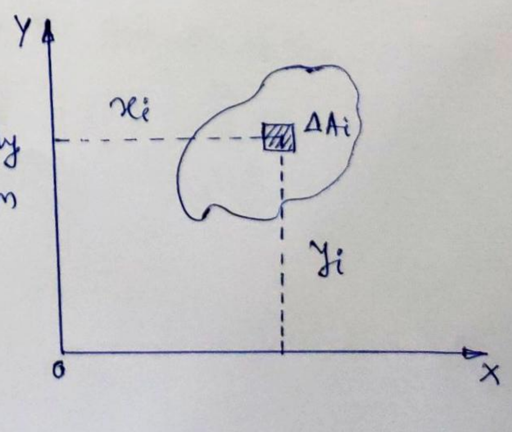
Center of Mass (C.M.)

It is the point where the entire mass of a body may be assumed to be concentrated.

Note The center of mass and the center of gravity of a body are different only when the gravitational field is not uniform and parallel. for most practical purposes they are assumed to be the same.

Analytical expressions of Centroid of 2-D Body:

Let us consider a plane area or a plane geometric figure of area A is placed with reference to any arbitarily chosen rectangular coordinate system as shown in fig. Now we choose a differential area ΔA_i within the figure. The coordinates of the middle point of the area with reference to the coordinate system are (x_i, y_i).



So the coordinates of the centroid of this area can be expressed in discrete form as

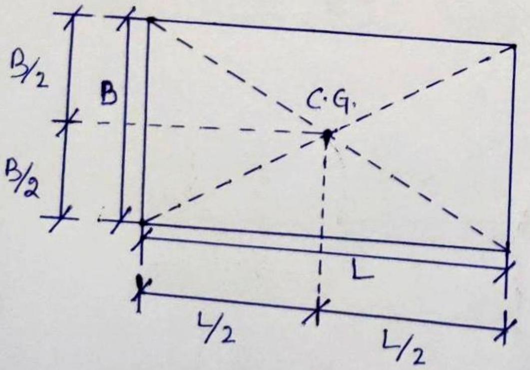
$$x_c = \frac{\sum_{i=1}^{n} \Delta A_i x_i}{\sum_{i=1}^{n} \Delta A_i} = \bar{x} \quad \text{and} \quad y_c = \frac{\sum_{i=1}^{n} \Delta A_i y_i}{\sum_{i=1}^{n} \Delta A_i} = \bar{y}$$

In continuous form they can be expressed as

$$x_c = \frac{\int_A x\,dA}{\int_A dA} = \bar{x} \quad \text{and} \quad y_c = \frac{\int_A y\,dA}{\int_A dA} = \bar{y}$$

In the above expression, we call the quantities $\int_A x\,dA$ and $\int_A y\,dA$ as first moment of area about y-axis and first moment of area about x-axis, respectively.

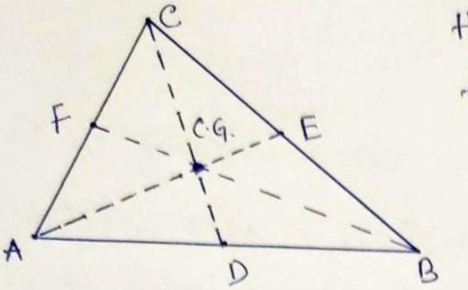
Centroid of a Rectangle



Here, the centriod is at that point where its diagonals meet each other. It is also a middle point of the length as well as the breadth of the rectangle.

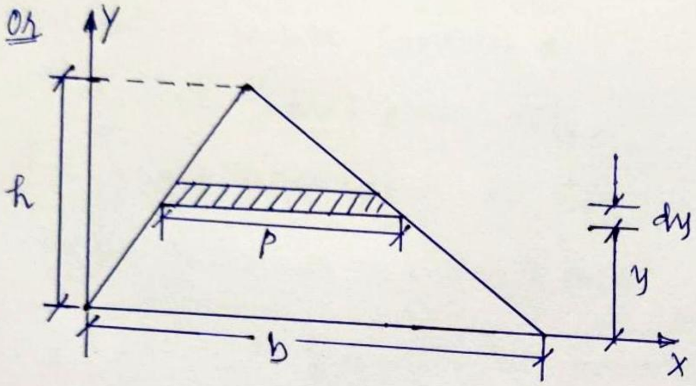
$$\bar{x} = x_c = \frac{L}{2} \; ; \; \bar{y} = y_c = \frac{B}{2}$$

Center of a Triangle :



Here, the centroid is that point where the three medians of the triangle meet as shown in fig.

Or



At a distance y from X-axis, a differential element is chosen of thickness dy. from the concept of similar triangles,

we can write, $\dfrac{p}{b} = \dfrac{h-y}{h}$ or $p = b\left[\dfrac{h-y}{h}\right]$

So, area of the differential shaded element

$$dA = p\,dy = b\left[\frac{h-y}{h}\right] dy$$

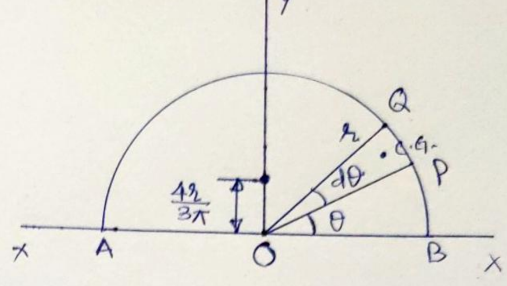
Hence $$\bar{y} = \frac{\int_0^h y \cdot b\left[\frac{h-y}{h}\right] dy}{\int_0^h b\left[\frac{h-y}{h}\right] dy} = \frac{b\int_0^h \left[y - \frac{y^2}{h}\right] dy}{b\int_0^h \left[1 - \frac{y}{h}\right] dy}$$

$$= \frac{\left[\frac{y^2}{2} - \frac{y^3}{3h}\right]_0^h}{\left[y - \frac{y^2}{2h}\right]_0^h} = \frac{\frac{h^2}{2} - \frac{h^3}{3h}}{h - \frac{h^2}{2h}} = \frac{\frac{h^2}{6}}{\frac{h}{2}}$$

$$\boxed{\bar{y} = y_c = \frac{h}{3}}$$

Centroid of a Semicircular Lamina:

Consider a semicircular lamina of radius r as shown in fig.



Let us determine the centroid of a plane area of semicircle for which polar coordinates are convenient to use. Consider an elemental radial area OPQ. The angle POQ being $d\theta$, this can be taken as triangle of area $\frac{1}{2} r \times d\theta . r = \frac{1}{2} r^2 d\theta$. The distance of the centroid of this elemental area is $\frac{2}{3} r$ from O. Hence the height of the centroid of the elemental area above AB is given by

$$\frac{2}{3} r \sin\theta$$

Therefore, the moment of the elemental area about AB is

$$\frac{1}{2} \times r^2 \times d\theta \times \frac{2}{3} \times r \sin\theta = \frac{1}{3} \times r^3 \sin\theta \, d\theta$$

The moment of whole area about AB is

$$\frac{1}{3} r^3 \int_0^{\pi} \sin\theta \, d\theta = \frac{1}{3} r^3 \times 2 \int_0^{\pi/2} \sin\theta \, d\theta$$

$$= \frac{2}{3} r^3$$

Let $\bar{y}$ be the height of the centroid above AB and we know the area of the semicircle $= \frac{\pi r^2}{2}$, then,

$$\frac{\pi r^2}{2} \bar{y} = \frac{2}{3} r^3 \iff \boxed{\bar{y} = \frac{4r}{3\pi}}$$

Therefore, the centroid of a semicircular area is located at $(\bar{x}, \bar{y})$ i.e. $\left(0, \frac{4r}{3\pi}\right)$.

Centroid of Composite figure:

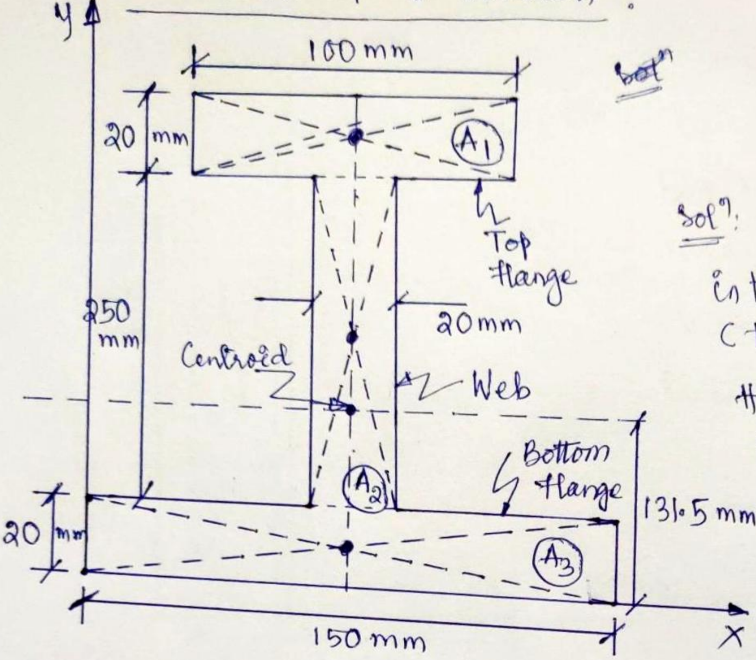
The following procedures are followed to find the centroid of a composite figure.

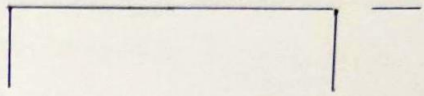
1. Divide the given composite figure into parts of known geometrical shapes such as rectangle, triangle, circle, semi-circle, quadrant of a circle.
2. Choose suitable reference axes at right angles to each other.
3. If the entire area of the composite figure lies in first quadrant, then both x distance and y distance will be positive. Otherwise, as per the sign convention of coordinates one has to consider.
4. If there is any line of symmetry, the centroid lies on this line.
5. If any part is removed or hole is made, its area should be treated as negative and should be substracted from the original area and, accordingly, the moments of these areas must be taken about reference axis.
6. Apply the principle of moments to find the values of $\bar{x} = \frac{\Sigma A_i x_i}{\Sigma A_i}$ and $\bar{y} = \frac{\Sigma A_i y_i}{\Sigma A_i}$.

Q.

Centroid of I-section :

~~Sol^n~~





Sol^n: Divide the composite figure in terms of the standard areas (three rectangles).

Here y-axis is the axis of symmetry, no need to find $\bar{x}$.

$\bar{x} = \underline{75\text{ mm}}$

A_1 = area of the top flange (mm²) $= 100 \times 20 = 2000\text{ mm}^2$

A_2 = area of the web (mm²) $= 250 \times 20 = 5000\text{ mm}^2$

A_3 = area of the bottom flange (mm²) $= 150 \times 20 = 3000\text{ mm}^2$

Centroidal distance y_i from x-axis

$$y_1 = 20 + 250 + \frac{20}{2} = 280\text{ mm}$$

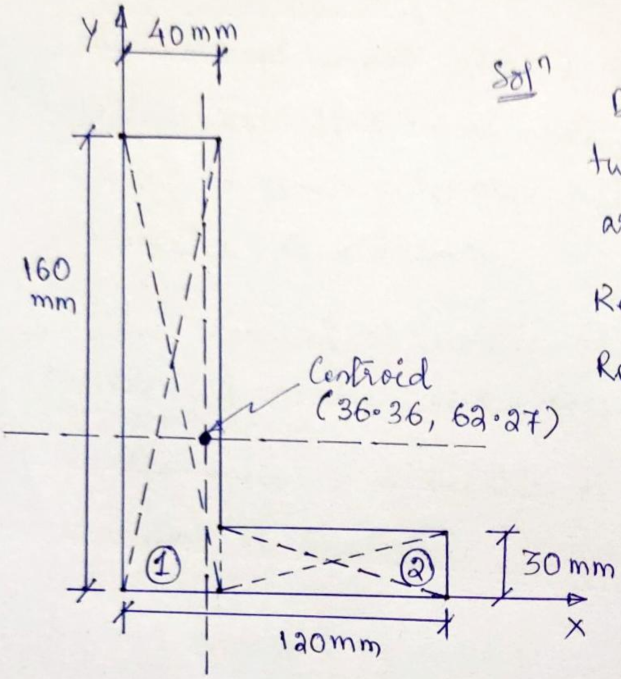
$$y_2 = 20 + \frac{250}{2} = 145\text{ mm}$$

$$y_3 = \frac{20}{2} = 10\text{ mm}$$

Component	Area (mm²)	Centroidal distance y_i from x-axis (mm)	Product $A_i y_i$ (mm³)
A_1	2000	280	560000
A_2	5000	145	725000
A_3	3000	10	30000
	$\Sigma A_i = 10000$		$\Sigma A_i y_i = 1315000$

$$\bar{y} = \frac{\Sigma A_i y_i}{\Sigma A_i} = \frac{1315000}{10000} = \underline{131.5\text{ mm}} \quad \text{Ans.}$$

Centroid of L-Section:



Sol^n

Divide the composite figure into two simple standard rectangular areas.

Rectangle 1, $A_1 = 160 \times 40 = 6400\ mm^2$

Rectangle 2, $A_2 = 80 \times 30 = 2400\ mm^2$

Component	Area (mm^2)	Centroidal distance x_i from y-axis (mm)	Centroidal distance y_i from y-axis (mm)	Product ($A_i x_i$) (mm^3)	Product ($A_i y_i$) (mm^3)
Rectangle 1	6400	20	80	128000	512000
Rectangle 2	2400	40+40=80	15	192000	36000
	$\Sigma A_i = 8800$			$\Sigma A_i x_i = 320000$	$\Sigma A_i y_i = 548000$

$$\bar{x} = \frac{\Sigma A_i x_i}{\Sigma A_i} = \frac{320000}{8800} = 36.36\ mm \quad \text{Ans.}$$

$$\bar{y} = \frac{\Sigma A_i y_i}{\Sigma A_i} = \frac{548000}{8800} = 62.27\ mm \quad \text{Ans.}$$

Moment of Inertia (M.I.)

The concept which gives a quantitative estimate of the relative distribution of area and mass of a body with respect to some reference axis is termed as the moment of inertia of the body.

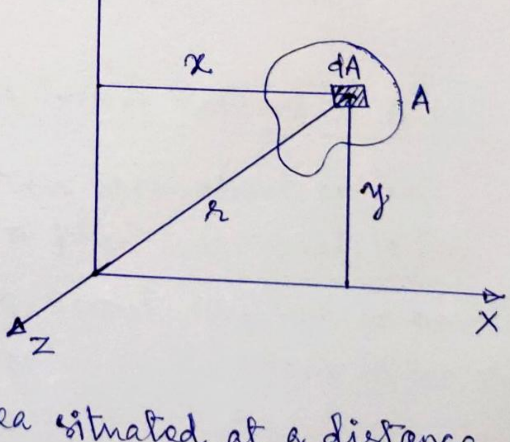
The moment of inertia of an area is called as the area moment of inertia or the second moment of area.

The moment of inertia of the mass is called as the mass moment of inertia.

Second moment of area :

Second moment of area is a property of a shape of a plane geometric figure which is used to predict its resistance to bending and deflection.

Consider a plane figure of area A in the X-Y plane as shown in fig.



Divide this area A into infinitesimal areas.

Let dA be any element of the area situated at a distance (x, y) from the axes.

Second moment of area with respect to x-axis $= I_x = \int y^2 \, dA$

Units: mm^4, cm^4 or m^4

Second moment of area with respect to y-axis $= I_y = \int x^2 \, dA$

Radius of Gyration :

The radius of gyration of an area is that distance from its moment of inertia axis at which the entire area could be considered as being concentrated without changing the numerical value of its moment of inertia. It is denoted by the symbol k.

$$\boxed{k = \sqrt{\frac{I}{A}}}$$

$I \rightarrow$ M.I. of the given section

$A \rightarrow$ Area of the section.

Radius of gyration about x-axis and y-axis will be expressed as

$$k_x = \sqrt{\frac{I_x}{A}} \quad \text{and} \quad k_y = \sqrt{\frac{I_y}{A}}$$

Unit

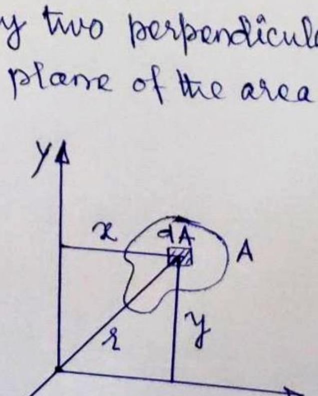
mm, cm or m

Note Radius of gyration depends not only on the shape of the area, but also on the position of reference.

Perpendicular Axis Theorem for Second Moment of Area :

The moment of inertia of a given area about an axis perpendicular to the plane through a point O is equal to the sum of the second moment of area about any two perpendicular axes through that point, the axes being in the plane of the area.

$$\boxed{I_z = I_x + I_y}$$



Proof:

Let dA be the elemental area at a distance r from O. $I_z = \int r^2\, dA = \int (x^2+y^2)\, dA$

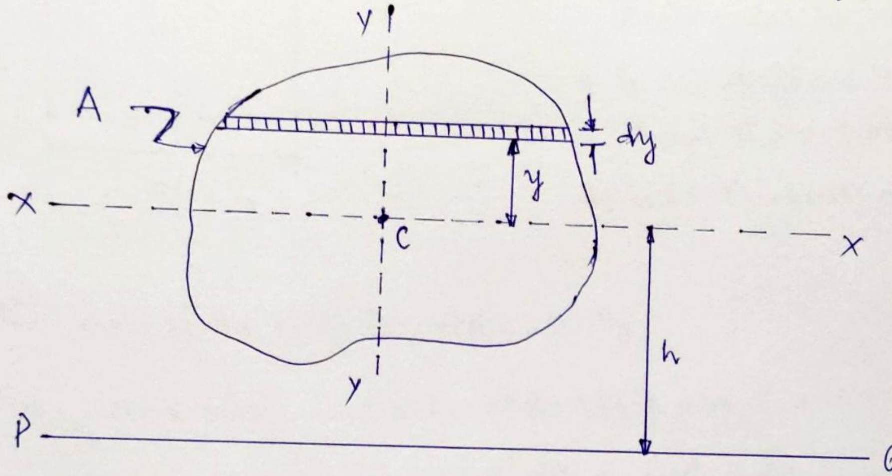
$= \int x^2 dA + \int y^2 dA$

$\Rightarrow \boxed{I_z = I_x + I_y}$ This parameter is also commonly called polar moment of inertia (J).

Parallel Axis Theorem for Second Moment of Area :

The second moment of area of any geometric figure with respect to any axis in its plane is equal to the second moment of area with respect to a parallel centroidal axis, plus the product of the total area and the square of the distance between the two parallel axes, one of which must be a centroidal axis.

Parallel axis theorem is also called Transfer axis theorem.



Proof: Consider an elementary strip of area dA at a distance of y from the centroidal axis x-x which is parallel to reference axis PQ, so that the distance of the strip from reference axis PQ is $(h+y)$.

Now, the second moment of area of the strip about the centroidal x-x axis is $y^2 dA$.

The second moment of area of the entire area A about the centroidal x-x axis is $\bar{I}_x = \int y^2 dA$

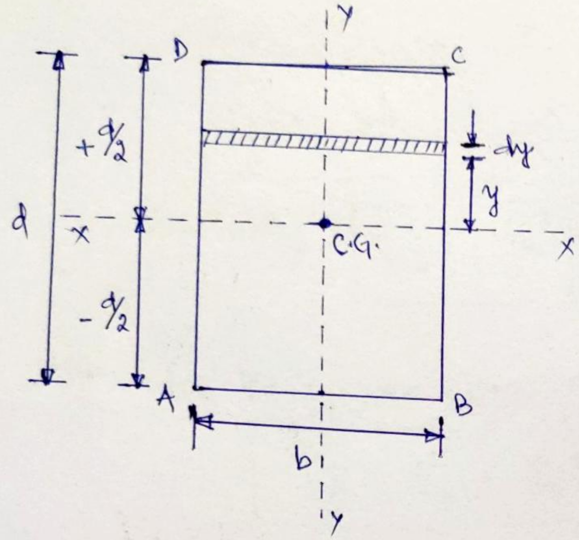
Similarly, second moment of area of the entire area about the reference axis PQ is ~~s~~ given by

$$I_{PQ} = \int (h+y)^2 dA = \int h^2 dA + \int y^2 dA + \int 2hy\,dA$$

or $I_{PQ} = Ah^2 + \bar{I}_x + 2h\int \cancel{y\,dA}^{\,0} \Leftrightarrow \boxed{I_{PQ} = \bar{I}_x + Ah^2}$

Second Moments of Area of Simple Areas

Rectangular Section :



Here ABCD is a rectangle, where
b is the width of the section
d is the depth of the section
dA is an area of the strip parallel to the base of the rectangular section
y is the distance of the strip from the x-x axis,
dy is the thickness of the strip.

The area of the strip is, $dA = b \cdot dy$

The second moment of area of the strip about x-x axis is

$$= \int y^2\, dA = \int y^2 \cdot b\, dy$$

Hence, the second moment of area of the whole section about the x-x axis is given by

$$\bar{I}_x = \int_{-d/2}^{+d/2} b \cdot y^2 \cdot dy = b \int_{-d/2}^{+d/2} y^2 dy = b \left[\frac{y^3}{3} \right]_{-d/2}^{+d/2}$$

$$\boxed{\bar{I}_x = \frac{bd^3}{12}}$$

Similarly, the second moment of area of the whole section about the y-y axis is given by

$$\boxed{\bar{I}_y = \frac{db^3}{12}}$$

Second Moment of Area about the Base of Rectangular Section

Applying the parallel axis theorem, we have,

$$I_{AB} = I_{base} = \bar{I}_x + A\left(\frac{d}{2}\right)^2 = \frac{bd^3}{12} + bd\left(\frac{d}{2}\right)^2$$

$$\boxed{I_{AB} = \frac{bd^3}{3}}$$

Similarly, $\boxed{I_{AD} = \frac{db^3}{3}}$

Note for the square section, $b = d$. Let $b = d = a$, then

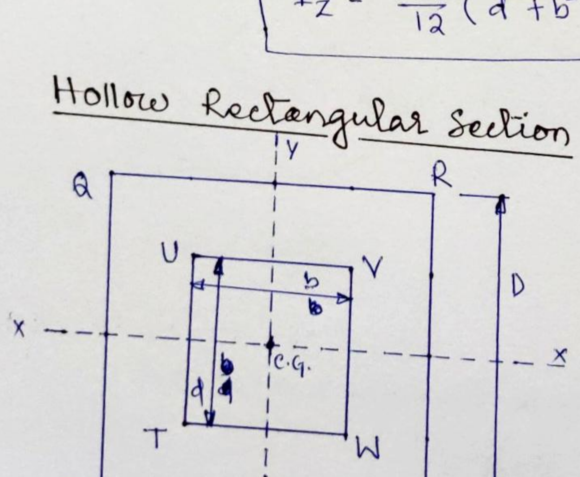
$$\boxed{\bar{I}_x = \frac{a^4}{12} = \bar{I}_y}$$

The polar moment of inertia can be obtained using the perpendicular axis theorem. i.e.

$$\bar{I}_z = \bar{I}_x + \bar{I}_y = \frac{bd^3}{12} + \frac{db^3}{12}$$

$$\boxed{\bar{I}_z = \frac{bd}{12}(d^2 + b^2)}$$

Hollow Rectangular Section



The second moment of area of the outer rectangle PQRS is given by

$$\bar{I}_x = \frac{BD^3}{12} \text{ and } \bar{I}_y = \frac{DB^3}{12}$$

The second moment of area of the inner rectangle TUVW is given by

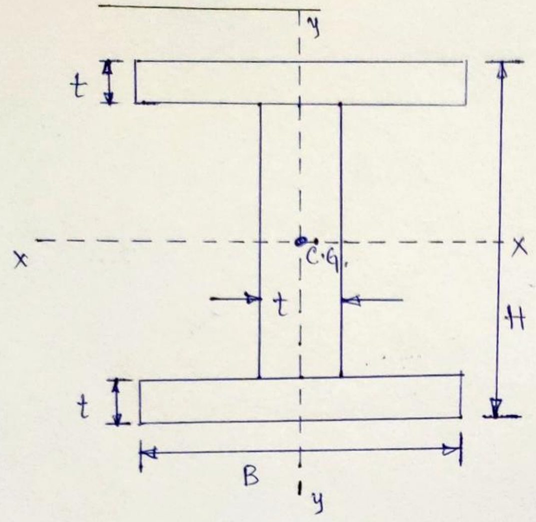
$$\bar{I}_x = \frac{bd^3}{12} \text{ and } \bar{I}_y = \frac{db^3}{12}$$

Thus, the second moment of area of the hollow rectangle is given by

$$\bar{I}_x = \frac{BD^3}{12} - \frac{bd^3}{12}$$

$$\bar{I}_y = \frac{DB^3}{12} - \frac{db^3}{12}$$

I-section



The I-section can be considered as a hollow rectangular section, where $B = B$, $D = H$ and

$$b = B - t, \quad d = H - 2t$$

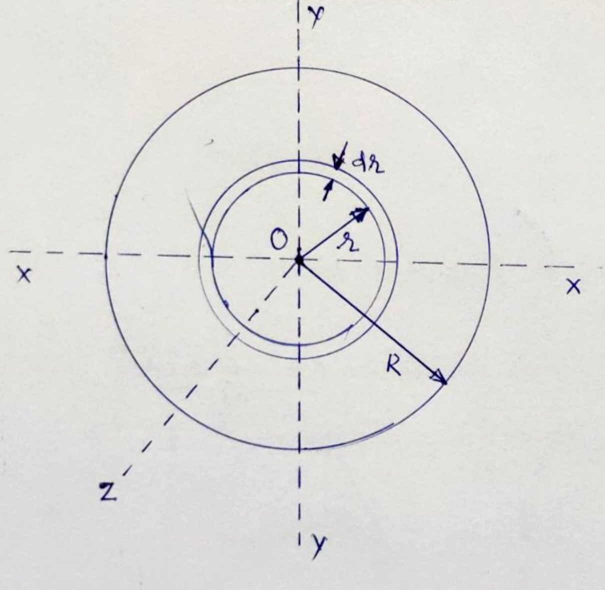
$$\bar{I}_x = \frac{BD^3}{12} - \frac{bd^3}{12}$$

$$\bar{I}_x = \frac{BH^3}{12} - \frac{(B-t)(H-2t)^3}{12}$$

$$\bar{I}_y = \frac{DB^3}{12} - \frac{db^3}{12}$$

$$\bar{I}_y = \frac{HB^3}{12} - \frac{(H-2t)(B-t)^3}{12}$$

Circular Section



Consider two mutually perpendicular axes x-x and y-y passing through the center of circle of radius R. Let the z-axis be perpendicular to the plane of circle and passing through O.

Now, consider an elementary ring of radius r and thickness dr.

The area of the ring is given by

$$dA = 2\pi r \cdot dr$$

The second moment of area of the ring about z-axis is

$$= \int r^2 \cdot 2\pi r \cdot dr$$

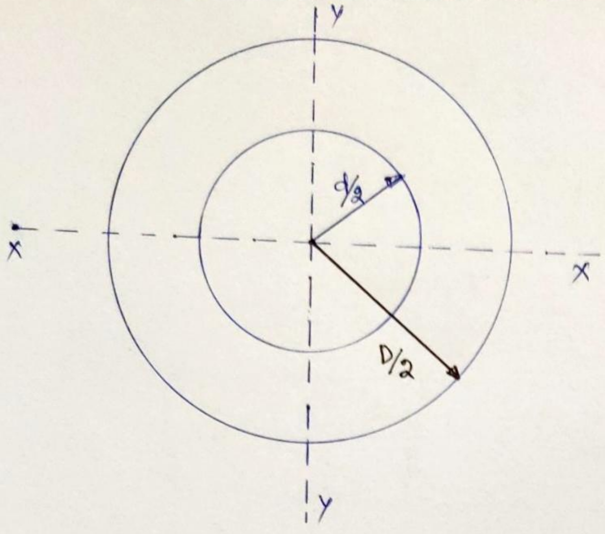
Thus, the second moment of area of the whole circular section about z-axis is given by

$$\bar{I}_z = J_0 = \int_0^R 2\pi r^3 \cdot dr = \frac{\pi R^4}{2} = \frac{\pi D^4}{32} \quad \left[\because R = \frac{D}{2}\right]$$

But $\bar{I}_z = \bar{I}_x + \bar{I}_y$ and for circular section $\bar{I}_x = \bar{I}_y$

$$\therefore \quad \bar{I}_x = \bar{I}_y = \frac{\pi D^4}{64}$$

Hollow Circular Section



Let us consider a hollow circular section as shown in fig.

The second moment of area about X-X axis for the outer circle is given by

$$\bar{I}_x = \frac{\pi D^4}{64}$$

Similarly, the second moment of area about X-X axis for inner circle is given by

$$\bar{I}_x = \frac{\pi d^4}{64}$$

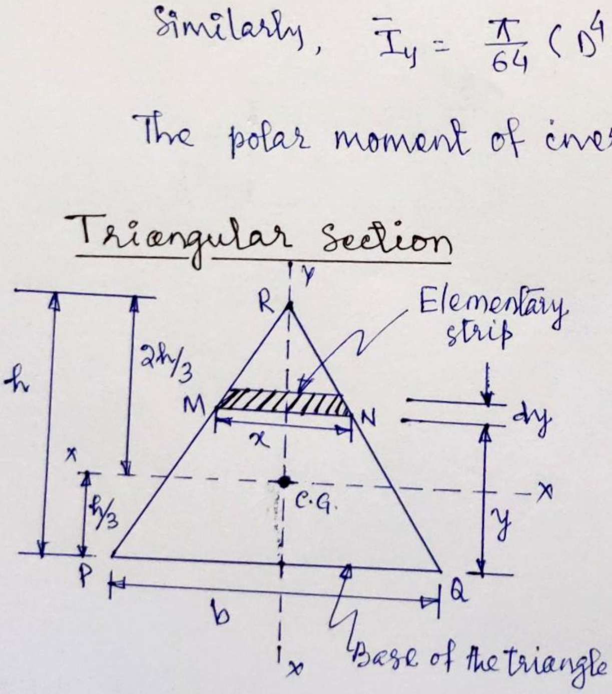
Hence, the second moment of area about x-x axis for the hollow circular section is given by

$$\bar{I}_x = \frac{\pi D^4}{64} - \frac{\pi d^4}{64} = \frac{\pi}{64}(D^4 - d^4)$$

Similarly, $\bar{I}_y = \frac{\pi}{64}(D^4 - d^4)$

The polar moment of inertia, $J_0 = \bar{I}_z = \frac{\pi}{32}(D^4 - d^4)$

Triangular Section



The area of the strip is given by

$$dA = x\,dy$$

The second moment of area of the elementary strip about an axis passing through the base PQ is

$$= \int y^2 dA = \int y^2 . x\,dy$$

from similar triangle, $\frac{x}{b} = \frac{h-y}{h} \Leftrightarrow x = b\left[\frac{h-y}{h}\right]$

Therefore, the second moment of area of the strip about the base is

$$= \int b\left(\frac{h-y}{h}\right) y^2 dy$$

The second moment of area of the whole area about the base is

$$= \int_0^h \frac{b}{h}(h-y)\,y^2\,dy = b\int_0^h \left(1-\frac{y}{h}\right) y^2\,dy$$

$$= b\int_0^h \left(y^2 - \frac{y^3}{h}\right) dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h$$

$$= b\left[\frac{h^3}{3} - \frac{h^4}{4h}\right]$$

$$\boxed{I_{PQ} = \frac{bh^3}{12} = I_{base}}$$

Now let us consider an axis passing through the centroid of the triangle and parallel to the base PQ.

Applying parallel axis theorem,

$$I_{PQ} = \bar{I}_x + Ah_1^2 \qquad \left[\because h_1 = \frac{h}{3}\right]$$

$$\Rightarrow \bar{I}_x = I_{PQ} - Ah_1^2 = \frac{bh^3}{12} - \left(\frac{1}{2}bh\right)\left(\frac{h}{3}\right)^2$$

$$\Rightarrow \boxed{\bar{I}_x = \frac{bh^3}{36}}$$